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M-Ideals and the "Basic Inequality"

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Two methods which can be used to show that, on a Banach space X, every bounded linear operator has a best compact approximation, namely the basic inequality method and the method of M-ideals, are shown to be basically equivalent. Thus, the paper responds to a question posed by S. Axler, I. D. Berg, N. Jewell, and A. Shields (1979, *Ann. of Math.* **109**, 601–612; 1980, *Trans. Amer. Math. Soc.* **261**, 159–167).

1. INTRODUCTION

The present note is devoted to the investigation of the relation between M-ideals of compact operators and Banach spaces satisfying the so-called "basic inequality," introduced by Axler, Berg, Jewell, and Shields [3, 4]. For a Banach space X to satisfy the basic inequality the following is required:

For all $S \in L(X)$, for all bounded nets $(A_\alpha) \subset L(X)$ such that $A_\alpha \to 0$ and $A_\alpha^* \to 0$ strongly and for all $\varepsilon > 0$ there is some index α_0 such that

$$\|S + A_{\alpha_0}\| \leqslant \max\{\|S\|, \|S\|_e + \|A_{\alpha_0}\|\} + \varepsilon. \tag{1}$$

(Here $L(X)$ denotes the space of all bounded linear operators, $K(X)$ the space of all compact operators, and $\|S\|_e$ the essential norm of S, i.e., the norm of the equivalence class $S + K(X)$ in the quotient space $L(X)/K(X)$.)

Axler, Berg, Jewell, and Shields introduced the basic inequality in order to show that on certain Banach spaces all bounded operators have best compact approximations. Their main result states:

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THEOREM 1.1. *If X satisfies the basic inequality and X^* enjoys the bounded approximation property, then $K(X)$ is proximinal in $L(X)$.*

They go on to show that l^p satisfies the basic inequality for $1<p<\infty$ as does c_0, whereas l^1, l^∞, and the L^p-spaces (for $p\neq 2$) fail the basic inequality. Therefore, $K(l^p)$ is proximinal in $L(l^p)$ for $1<p<\infty$, a result which can also be derived from the fact that $K(l^p)$ is an M-ideal in $L(l^p)$ for these p (see, for instance, [26]). Apart from the papers quoted in [3], the problem of best compact approximation is considered, e.g., in [2, 5, 9, 10, 13, 14, 17, 18, 32, 33, 37].

Recall that a closed subspace J of a Banach space X is called an M-ideal if there is a linear projection P from X^* onto $J^\perp$, the annihilator of J in X^*, satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \forall x^* \in X^*.$$

This notion is due to Alfsen and Effros [1] and is studied in detail in [6] and our monograph [21]. A result from M-ideal theory of particular interest in the present context states that M-ideals are proximinal (see [1, Cor. I.5.6; 6, Prop. 6.5; 21, Prop. II.1.1], and see also below). In due course we shall need the following characterisation of M-ideals in terms of the so-called three-ball property: J is an M-ideal in X if and only if for all $x\in B_X$ (the closed unit ball of X), $y_1, y_2, y_3 \in B_J$, and $\varepsilon>0$ there is some $y\in J$ such that $\|y_i + x - y\| \leqslant 1+\varepsilon$ $(i=1, 2, 3)$. For a proof we refer to [25, Th. 6.17] or [21, Th. I.2.2].

Moreover, the authors of [3] obtain the proximinality of $H^\infty + C(\mathbb{T})$ in $L^\infty(\mathbb{T})$ by basic inequality techniques; and Luecking [29] gives a proof of the same result using M-ideal methods, viz. by showing that $(H^\infty + C(\mathbb{T}))/H^\infty$ is an M-ideal in $L^\infty(\mathbb{T})/H^\infty$. Also, Davidson and Power obtain theorems on best approximation both by M-ideal methods and by basic inequality techniques [12].

These similarities suggest that there might be a close relation between the two methods. The purpose of this paper is to point this out. More precisely, we show that a revised version of the basic inequality is in fact equivalent to $K(X)$ being an M-ideal in $L(X)$. But we also note that the basic inequality as it stands and the M-ideal property are unrelated.

2. A CHARACTERISATION OF M-IDEALS BY A REVISED BASIC INEQUALITY

Before giving this characterisation, we first show that the basic inequality (1) does not imply that $K(X)$ is an M-ideal in $L(X)$, and that the converse does not hold, either.

EXAMPLE 2.1. *Every subspace X of c_0 satisfies the basic inequality.*

Proof. Let S, A_α, and $\varepsilon > 0$ be as in the definition of the basic inequality, and let (P_n) denote the sequence of coordinate projections on c_0. For a compact operator K on X we have

$$\begin{aligned}\|(\mathrm{Id} - P_n)S\| &\leqslant \|(\mathrm{Id} - P_n)(S-K)\| + \|(\mathrm{Id} - P_n)K\| \\ &\leqslant \|S-K\| + \|(\mathrm{Id} - P_n)K\|\end{aligned}$$

so that for some m

$$\|(\mathrm{Id} - P_m)S\| \leqslant \|S\|_e + \varepsilon,$$

since $P_n \to \mathrm{Id}$ uniformly on the relatively compact subset $K(B_X)$ of c_0. Fixing this m we obtain from the strong convergence of (A_α^*)

$$\|P_m A_{\alpha_0}\| = \|A_{\alpha_0}^* P_m^*\| \leqslant \varepsilon$$

for some α_0. Altogether this yields

$$\begin{aligned}\|S + A_{\alpha_0}\| &= \max\{\|P_m(S + A_{\alpha_0})\|, \|(\mathrm{Id} - P_m)(S + A_{\alpha_0})\|\} \\ &\leqslant \max\{\|S\| + \varepsilon, \|S\|_e + \varepsilon + \|A_{\alpha_0}\|\}. \quad \blacksquare\end{aligned}$$

In fact, a faithful rewording of the proof of Theorem 2 in [3] yields that every subspace of l^p $(1 < p < \infty)$ and more generally every subspace of an (M_p)-space (in the sense of [30]) satisfies the basic inequality.

To obtain the desired counterexample it remains to quote, e.g., from [28, p. 111] that there are subspaces X of c_0 without the metric compact approximation property. In particular, such a space X satisfies the basic inequality without $K(X)$ being an M-ideal in $L(X)$, since the latter property is known to imply the metric compact approximation property [20]. Still, one can show for subspaces X of l^p or c_0 that $K(X)$ is an M-ideal in some subspace of $L(X)$.

Next we give an example to show that the converse implication does not hold either.

EXAMPLE 2.2. *There is a Banach space X failing the basic inequality such that $K(X)$ is an M-ideal in $L(X)$.*

Proof. Consider a reflexive Orlicz sequence space h_M which contains isomorphic copies of l^p for two different values of p. To be more specific, let $M(t) = t^{3 + \sin(\log|\log t|)}$. Then h_M is reflexive and contains copies of l^2 and l^p for some $p < 2$, which is fixed in the following. Note that l^2 embeds into $(h_M)^*$ as well; for these matters see [27, 4.b.3, 4.c.2]. N. Kalton [24] (see also [23]) has shown that there is an equivalent norm on h_M such that the renormed Orlicz space, X, has the following properties:

• $K(X)$ is an M-ideal in $L(X)$.

• For every $\delta > 0$, X contains a subspace F_δ whose Banach–Mazur distance to l^p is $d(F_\delta, l^p) < 1 + \delta$.

• For every $\delta > 0$, X^* contains a subspace H_δ with $d(H_\delta, l^2) < 1 + \delta$. Consequently X admits of a quotient E_δ such that $d(E_\delta, l^2) < 1 + \delta$.

Now, the basic inequality can be formulated verbatim for operators acting between two different Banach spaces. This point of view is adopted in [17, 18]. It is noted there that the basic inequality holds for operators from quotients of X to subspaces of X once it holds for operators from X to X. Therefore, should the basic inequality hold for the above space X, it would hold for operators from E_δ to F_δ. Letting δ tend to 0 we deduce that the basic inequality holds for operators from l^2 to l^p, which is not the case as is calculated in [18] (recall $p < 2$). ∎

Despite these examples we show that the basic inequality method of [3] and the M-ideal method for obtaining best compact approximations are basically equivalent. To achieve this we propose to revise the basic inequality in such a way that the existence assumption which is inherent in Theorem 1.1 under the form of the approximation property becomes part of a "revised basic inequality." (The problem with the original basic inequality is the "$\forall(A_\alpha)$" quantifier, which is too much to be expected.) First, we present a general proposition along these lines. Note that under the assumption of Proposition2.3, $\operatorname{ran}(\pi)$ is canonically isometric to J^* so that it makes sense to consider the $\sigma(X, J^*)$-topology.

PROPOSITION 2.3. *Let J be a subspace of a Banach space X such that $J^\perp$ is the kernel of a contractive linear projection π. Then the following assertions are equivalent:*

(i) *J is an M-ideal in X.*

(ii) *For all $x \in X$ there is a net (y_α) in J such that $y_\alpha \to x$ in the $\sigma(X, J^*)$-topology and*

$$\limsup \|z + (x - y_\alpha)\| \leqslant \max\{\|z\|, \|z + J\| + \|x\|\} \qquad \forall z \in X.$$

(iii) *For all $x \in B_X$ there is a net (y_α) in J such that $y_\alpha \to x$ in the $\sigma(X, J^*)$-topology and*

$$\limsup \|y + (x - y_\alpha)\| \leqslant 1 \qquad \forall y \in B_J.$$

The proof of Proposition 2.3 depends on the following version of the principle of local reflexivity.

LEMMA 2.4. *Let X be a Banach space and $J \subset X$ a closed subspace. Suppose that $E \subset X^{**}$ and $F \subset X^*$ are finite dimensional subspaces and that $\varepsilon > 0$ is given. Then there is a linear operator $T: E \to X$ such that*

- $\langle Tx^{**}, x^* \rangle = \langle x^{**}, x^* \rangle \; \forall x^{**} \in E,\, x^* \in F,$
- $(1-\varepsilon)\,\|x^{**}\| \leqslant \|Tx^{**}\| \leqslant (1+\varepsilon)\,\|x^{**}\| \; \forall x^{**} \in E,$
- $T|_{E \cap X} = \mathrm{Id},$
- $T(E \cap J^{\perp\perp}) \subset J.$

Proof. This is a special case of [7, Th. 3.2], but the lemma also follows from [8] since J and $E \cap X$ are easily seen to form a "friendly collection" in the terminology of that paper. ▌

Proof of Proposition 2.3. (i) $\Rightarrow$ (ii): Let Q denote the M-projection from X^{**} onto $J^{\perp\perp}$. Then

$$\begin{aligned} \|z + x - Qx\| &= \|Qz + (\mathrm{Id} - Q)(z - Qz + x)\| \\ &= \max\{\|z\|, \|z - Qz + x\|\} \\ &\leqslant \max\{\|z\|, \|z + J\| + \|x\|\} =: \mu, \end{aligned}$$

since $\ker Q \cong X^{**}/J^{\perp\perp}$ and $\|z + J^{\perp\perp}\| = \|z + J\|$ for $z \in X$.

Now consider the set $\mathbf{A}$ of all triples $\alpha = (E, F, \varepsilon)$, where $E \subset X^{**}$ and $F \subset X^*$ are finite dimensional subspaces and $\varepsilon > 0$. Then $\mathbf{A}$ is directed in a natural way. We denote by T_α a local reflexivity operator with the properties spelt out in Lemma 2.4 and define $y_\alpha = T_\alpha(Qx)$. (Note that y_α is eventually defined.) Then $y_\alpha \to x$ in the desired fashion. To see this observe that the copy of J^* we are considering coincides with the preannihilator of $\ker Q$, and hence we obtain for finite dimensional subspaces $F \subset J^*$, $y^* \in F$, and sufficiently large α

$$\langle y_\alpha, y^* \rangle = \langle T_\alpha(Qx), y^* \rangle = \langle Qx, y^* \rangle = \langle x, y^* \rangle.$$

Finally, we obtain (again, the term involving T_α is eventually defined)

$$\|z + x - y_\alpha\| = \|T_\alpha(z + x - Qx)\| \leqslant \|T_\alpha\|\, \mu,$$

which yields the desired conclusion.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): This follows from [35, Cor. 1.2]. ▌

Note that the net gained in the preceding proposition is necessarily bounded, since the choice $z = 0$ yields $\limsup \|y_\alpha - x\| \leqslant \|x\|$.

3. APPLICATIONS

We now discuss special instances where Proposition 2.3 applies. In the case where $X = J^{**}$ this proposition is somewhat stronger than [19, Prop. 4.3]. Consider now the subspace $K(X)$ of $L(X)$. It is known that $K(X)^{\perp}$ is the kernel of a contractive projection if X has the metric compact approximation property [22].

THEOREM 3.1. *For a Banach space X, the following assertions are equivalent*:

(i) *$K(X)$ is an M-ideal in $L(X)$.*

(ii) *For all $T \in L(X)$ there is a net (K_α) in $K(X)$ such that $K_\alpha^* \to T^*$ strongly and*

$$\limsup \|S + T - K_\alpha\| \leqslant \max\{\|S\|, \|S\|_e + \|T\|\} \qquad \forall S \in L(X). \tag{2}$$

(iii) *For all $T \in L(X)$ there is a net (K_α) in $K(X)$ such that $K_\alpha^* \to T^*$ strongly and*

$$\limsup \|S + T - K_\alpha\| \leqslant \max\{\|S\|, \|T\|\} \qquad \forall S \in K(X).$$

Proof. (i) $\Rightarrow$ (ii): Since the functionals $u \mapsto \langle u^{**}x^{**}, x^* \rangle$ belong to the copy of $K(X)^*$ in $L(X)^*$, there is, by Proposition 2.3, a net (L_α) in $K(X)$ such that $L_\alpha^* \to T^*$ in the weak operator topology satisfying (2). Now (2) will not be spoiled by taking convex combinations of the L_α. Hence, there are $K_\alpha \in \operatorname{co}\{L_\beta \mid \beta \geqslant \alpha\}$ with $K_\alpha^* \to T^*$ strongly fulfilling inequality (2).

(ii) $\Rightarrow$ (iii): This is obvious, again.

(iii) $\Rightarrow$ (i): Clearly, condition (iii) implies the three-ball property so that $K(X)$ is an M-ideal in $L(X)$. ∎

It is worthwhile mentioning that Theorem 3.1 extends verbatim to operators acting between distinct Banach spaces.

An application of (2) with $S = 0$ shows that $\limsup \|T - K_\alpha\| \leqslant \|T\|$. Therefore there is some reason to call condition (ii) a "revised basic inequality" for X. As a matter of fact, apart from a superficial resemblence of (ii) with the basic inequality, (2) constitutes the core of the proof of Theorem 1.1, as an inspection of the argument in [3] shows; all the results of that paper can be obtained on the basis of the revised basic inequality. This observation enables us to provide an approximation theoretic result for M-ideals of compact operators which is more precise than their mere proximinality.

COROLLARY 3.2. *Suppose $K(X)$ is an M-ideal in $L(X)$. Let $T \in L(X)$ and* *let* $(T_\alpha)_{\alpha \in \mathbf{A}} \subset K(X)$ *be a bounded net such that* $T_\alpha^* \to T^*$ *strongly. Then there exists some* $K \in \overline{\mathrm{co}}\{T_\alpha \mid \alpha \in \mathbf{A}\}$ *such that* $\|T-K\| = \|T\|_e$.

Proof. If (K_α) denotes a net devised by Theorem 3.1(ii), then $T_\alpha^* - K_\alpha^* \to 0$ in the weak operator topology and hence $T_\alpha - K_\alpha \to 0$ in the weak topology $\sigma(K(X), K(X)^*)$ (cf. [34]). Therefore, $\|\hat{T}_\alpha - \hat{K}_\alpha\| \to 0$ for some $\hat{T}_\alpha \in \mathrm{co}\{T_\beta \mid \beta \geqslant \alpha\}$, $\hat{K}_\alpha \in \mathrm{co}\{K_\beta \mid \beta \geqslant \alpha\}$. Since $(\hat{K}_\alpha)$ still satisfies (2), so does $(\hat{T}_\alpha)$, and by the argument leading to Theorem 1 in [3], T has a best compact approximant $K \in \overline{\mathrm{co}}\{\hat{T}_\alpha \mid \alpha \in \mathbf{A}\} = \overline{\mathrm{co}}\{T_\alpha \mid \alpha \in \mathbf{A}\}$. ∎

A similar result holds for best approximations by elements of general M-ideals: If J is an M-ideal in X and $x \in X$, and if (y_α) is a net in J converging to x in the $\sigma(X, J^*)$-topology, then there exists some $y \in \overline{\mathrm{co}}\{y_\alpha \mid \alpha \in \mathbf{A}\}$ such that $\|x-y\| = \|x+J\|$. This can be seen as above.

In the next result, which is a corollary to one of the main theorems in [36], we give a more precise version of condition (ii) of Theorem 3.1.

PROPOSITION 3.3. *For a Banach space X, the following assertions are equivalent*:

(i) *$K(X)$ is an M-ideal in $L(X)$.*

(ii) *There exists a net (K_α) in $K(X)$ such that $K_\alpha^* \to \mathrm{Id}_{X^*}$ strongly and*

$$\limsup \|K_\alpha S + (\mathrm{Id} - K_\alpha) T\| \leqslant \max\{\|S\|, \|T\|\} \qquad \forall S, T \in L(X).$$

(iii) *There exists a net (K_α) in $K(X)$ such that $K_\alpha^* \to \mathrm{Id}_{X^*}$ strongly and*

$$\limsup \|S + (\mathrm{Id} - K_\alpha) T\| \leqslant \max\{\|S\|, \|S\|_e + \|T\|\} \qquad \forall S, T \in L(X).$$

(iv) *There exists a net (K_α) in $K(X)$ such that $K_\alpha^* \to \mathrm{Id}_{X^*}$ strongly and*

$$\limsup \|S + (\mathrm{Id} - K_\alpha) T\| \leqslant \max\{\|S\|, \|T\|\} \qquad \forall S \in K(X), \quad T \in L(X).$$

Proof. (i) $\Rightarrow$ (ii): This is proved in [36, Theorem 5.2].

(ii) $\Rightarrow$ (iii): By a convex combinations argument we may suppose that $K_\alpha \to \mathrm{Id}$ strongly, too. For $\varepsilon > 0$ fix α such that $\|(\mathrm{Id} - K_\alpha) S\| \leqslant \|S\|_e + \varepsilon$. This is possible by the same argument as that in Example 2.1. Then fix β_0 such that $\|K_\beta K_\alpha - K_\alpha\| \leqslant \varepsilon$, hence $\|(\mathrm{Id} - K_\beta)(\mathrm{Id} - K_\alpha) - (\mathrm{Id} - K_\beta)\| \leqslant \varepsilon$ for $\beta \geqslant \beta_0$. Consequently

$$\begin{aligned} \|S + (\mathrm{Id} - K_\beta) T\| &= \|K_\beta S + (\mathrm{Id} - K_\beta)(S + T)\| \\ &\leqslant \|K_\beta S + (\mathrm{Id} - K_\beta)((\mathrm{Id} - K_\alpha) S + T)\| + \varepsilon \|S\| \\ &\leqslant \max\{\|S\|, \|(\mathrm{Id} - K_\alpha) S + T\|\} + \varepsilon + \varepsilon \|S\| \\ &\leqslant \max\{\|S\|, \|S\|_e + \|T\|\} + \varepsilon(\|S\| + 2) \end{aligned}$$

for large enough β, which yields our claim.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): This is known from [36] or [24] and follows from Theorem 3.1 as well. ▮

The coordinate projections on l^p do not satisfy the inequality in (ii), as was pointed out in [36]. However, the do work in (iii), since l^p satisfies the original basic inequality, and hence in (iv), too; the latter follows also from the proof of [24, Th. 2.4].

Our final aim is to apply the ideas of the present section to nest algebras. Let H denote "a" separable complex infinite dimensional Hilbert space. For convenience we put $\mathscr{K} = K(H)$, $\mathscr{L} = L(H)$. A *nest* $\mathscr{N}$ is a strongly closed totally ordered set of projections on H containing 0 and Id. The corresponding *nest algebra* $\mathscr{A} = \mathscr{A}(\mathscr{N})$ consists of all those operators on H that leave ran(P) invariant for each $P \in \mathscr{N}$. All the results on nest algebras used below can be found in the survey articles [11, 31].

Feeman [15] discusses a property Δ, reminiscent of the basic inequality, that a subspace of $L(H)$ might or might not have. (We omit the definition.) He shows for a nest algebra $\mathscr{A}$ with property Δ that $\mathscr{A} + \mathscr{K}$ (which is closed) is proximinal in $L(H)$. However, he is able to check Δ only for the nest consisting of the coordinate projections with respect to some fixed orthonormal basis of H. In [16] he obtains proximinality of $\mathscr{A} + \mathscr{K}$ for every nest algebra $\mathscr{A}$ in that he proves that $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is an M-ideal in $\mathscr{L}/\mathscr{A}$. Another proof of this fact is contained in [12]. Since $\mathscr{L}/\mathscr{A}$ happens to be the bidual of $(\mathscr{A} + \mathscr{K})/\mathscr{A} \cong \mathscr{K}/(\mathscr{A} \cap \mathscr{K})$, this result also follows from the stability of the class of M-embedded spaces with respect to quotients [20].

It is asked in [15] which nest algebras have property Δ. The following proposition states that all nest algebras enjoy a "revised property Δ." We remark in passing that an analogous result can be proved for the subalgebra $\mathscr{A}$ of $L(l^p)$ consisting of those operators which have an upper triangular matrix representation with respect to the canonical basis of l^p, for $1 < p < \infty$. This answers another question posed in [15].

PROPOSITION 3.4. *Let $\mathscr{A}$ be a nest algebra. Then for every $T \in L(H)$ there is a sequence (T_n) in $\mathscr{A} + \mathscr{K}$ such that $T_n \to T$ strongly and ($d(\cdot, \mathscr{A})$ denoting distance to $\mathscr{A}$)*

$$\limsup d(S + T - T_n, \mathscr{A}) \leqslant \max\{d(S, \mathscr{A}), d(S, \mathscr{A} + \mathscr{K}) + d(T, \mathscr{A})\} \qquad \forall S \in L(H).$$

Proof. If we let $X = \mathscr{L}/\mathscr{A}$ and $J = (\mathscr{A} + \mathscr{K})/\mathscr{A}$, then the desired inequality, in a version for nets, reads

$$\limsup \|S + T - T_\alpha\|_X \leqslant \max\{\|S\|_X, \|S\|_{X/J} + \|T\|_X\} \qquad \forall S \in L(H),$$

where $\|S\|_X$ denotes the norm of the equivalence class of S in X, etc. Since J is an M-ideal in X by the above discussion this is fulfilled for some bounded net of equivalence classes $([T_\alpha])$ in J tending to $[T]$ in the $\sigma(X, J^*)$-sense, by Proposition 2.3.

It remains to investigate the convergence of this net, which is just the weak* convergence in $X \cong J^{**}$. If (T_α) is a bounded net of compact representatives of the $[T_\alpha] \in (\mathscr{A} + \mathscr{K})/\mathscr{A} \cong \mathscr{K}/(\mathscr{A} \cap \mathscr{K})$ and R is a $\sigma(L(H), K(H)^*)$-limit point, then $R - T \in (\mathscr{A} \cap \mathscr{K})^{\perp\perp} = \mathscr{A}$. Thus, upon replacing (T_α) by an appropriate subnet of $(T_\alpha - R)$ we may assume that $T_\alpha \to T$ in the weak operator topology. Therefore we may even assume that $T_\alpha \to T$ strongly by a convex combinations argument. (The point here is that a linear functional on $L(H)$ is continuous for the strong operator topology if and only if it is continuous for the weak operator topology so that a convex subset of $L(H)$ is strongly closed if and only if it is weak operator closed.) Since the strong operator topology is metrizable on bounded sets, we may pick a subsequence of (T_α), say (T_n), with all the desired properties. This completes the proof of Proposition 3.4. ∎

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